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GENERALIZED GRUNSKY COEFFICIENTS AND INEQUALITIES

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ABSTRACT

The generalized Grunsky coefficients are defined in this paper for all locally univalent meromorphic functions in any domain in the complete complex plane. Various explicit formulas for these coefficients are established. Necessary conditions for univalence are obtained in arbitrary domains and in the unit disc in particular. The first one generalizes Grunsky inequalities and the second one is an extension of the Nehari–Schwarzian derivative condition.

Introduction

For a sequence of complex numbers $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ and an integer l, we formally denote

$$\alpha(w)^l = \left(\sum_{n=1}^{\infty} \alpha_n w^n\right)^l = \sum_{k=l}^{\infty} A_{k,l}(\alpha) w^k = \sum_{k=l}^{\infty} A_{k,l}(\alpha_1, \alpha_2, \ldots) w^k.$$

The coefficients $A_{k,l}(\alpha)$ are the so-called Bell polynomials (cf. Comtet [5] chapter 5.4). Hummel [9] and Todorov [12] found explicit formulas for Grunsky coefficients of a function $F(t) \in \Sigma$ in terms of Bell polynomials of the sequence

$$\alpha_n = \frac{f^{(n)}(0)}{n!} \quad \text{where } f(z) = F(1/z)^{-1}, \qquad n \geqq 1.$$

The purpose of this paper is to study a generalization (due to Aharonov [3]) of Grunsky coefficients, derive their explicit formulas in terms of Bell polynomials of some sequences and deduce necessary conditions for univalence generalizing Grunsky inequalities to arbitrary domains on the one hand and the Schwarzian derivative condition in the unit disc on the other hand. These results are derived

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in the last two sections. In order to obtain these results we start with a brief review on some elementary properties of Bell polynomials and then derive explicit formulas for Aharonov invariants and for another sequence of invariants. In particular two formulas for Bernoulli numbers are derived in the second section.

1. Bell polynomials

Most of the results in this section are known, but for the sake of completeness we briefly sketch proofs for some of them.

EXAMPLE 1. Using the expansion

$$\left[\frac{w}{(1-w)^{a+1}}\right]^l = \sum_{k=l}^{\infty} \binom{k+al-1}{k-l} w^k, \qquad a \geqq 0$$

one gets

$$A_{k,l}\left(\left\{\binom{n+a-1}{n-1}\right\}_{n=1}^{\infty}\right) = \binom{k+al-1}{k-l}, \tag{1.1}$$

e.g.,

$$A_{k,l}(1,1,\dots) = \binom{k-1}{l-1} \qquad (\text{for } a=0)$$

and

$$A_{k,l}(1,2,\dots) = \binom{k+l-1}{k-l} \qquad (\text{for } a=1).$$

EXAMPLE 2. Taking $\alpha_n = 1/(n-1)!$, $\alpha(w) = we^w$ and

$$\alpha(w)^m = w^m e^{mw} = \sum_{k=m}^{\infty} \frac{m^{k-m}}{(k-m)!} w^k$$

we deduce that

$$A_{k,m}\left(\left\{\frac{1}{(n-1)!}\right\}_{n=1}^{\infty}\right) = \frac{m^{k-m}}{(k-m)!}. \tag{1.2}$$

LEMMA A. *For a natural number l an explicit formula for $A_{k,l}(\alpha)$ is* (see [5] and [12]):

$$A_{k,l}(\alpha) = \sum \frac{l!}{\nu_1!\,\nu_2!\dots\nu_s!}\,\alpha_1^{\nu_1}\alpha_2^{\nu_2}\dots\alpha_s^{\nu_s}, \qquad k \geqq l \geqq 0, \tag{1.3}$$

the sum taken over all the nonnegative ν_n's satisfying

$$\nu_1+\nu_2+\cdots+\nu_s=l, \qquad \nu_1+2\nu_2+\cdots+s\nu_s=k$$

so that in particular $s \leqq k-l+1$.

EXAMPLE 3.

$$\begin{cases} A_{k,1}(\alpha)=\alpha_k, \qquad A_{k,k}(\alpha)=\alpha_1^k, \\ A_{k+1,k}(\alpha)=k\alpha_1^{k-1}\alpha_2, \\ A_{k+2,k}(\alpha)=\binom{k}{2}\alpha_1^{k-2}\alpha_2^2+\binom{k}{1}\alpha_1^{k-1}\alpha_3. \end{cases} \tag{1.4}$$

LEMMA B. (i) *Bell polynomials* $A_{k,l}(\alpha)$ *are homogeneous of degree* l:

$$A_{k,l}(t\alpha_1, t\alpha_2,\ldots)=t^l A_{k,l}(\alpha_1,\alpha_2,\ldots) \tag{1.5}$$

and of weight k:

$$A_{k,l}(t\alpha_1, t^2\alpha_2,\ldots)=t^k A_{k,l}(\alpha,\alpha_2,\ldots). \tag{1.6}$$

(ii) *Bell polynomials of a sequence* $\{\alpha_n\}_{n=1}^{\infty}=\alpha$ *satisfy*

$$A_{k,l}(\alpha)=\sum_{n=l-m}^{k-m} A_{k-n,m}(\alpha)A_{n,l-m}(\alpha), \qquad k\geqq l, \quad -\infty<m<\infty; \tag{1.7}$$

in particular, for $m=1$ *we have*

$$A_{k,l}(\alpha)=\sum_{n=1}^{k-l+1} \alpha_n A_{k-n,l-1}(\alpha). \tag{1.7'}$$

We also have

$$\frac{k}{l}A_{k,l}(\alpha)=\sum_{n=1}^{k-l+1} n\alpha_n A_{k-n,l-1}(\alpha), \qquad l\neq 0. \tag{1.8}$$

(iii) *Applying* (1.7) *to Example* 1 *one gets*

$$\sum_{n=0}^{k-l}\binom{a+n}{n}\binom{k-n}{l}=\binom{a+k+1}{k-l}, \qquad k\geqq l\geqq 1, \quad a\geqq 0. \tag{1.9}$$

LEMMA C. *For any pair of sequences* $\alpha=\{\alpha_n\}$, $\beta=\{\beta_n\}$, *the binomial formula yields* (see [5] Section 3.3)

$$A_{k,l}(\alpha\pm\beta)=\sum_{m=0}^{l}(\pm 1)^{l-m}\binom{l}{m}\sum_{n=m}^{m+k-l} A_{n,m}(\alpha)A_{k-n,l-m}(\beta). \tag{1.10}$$

In particular if $\alpha = \{0, \alpha_2, \alpha_3, \dots\}$, $\beta = \{\alpha_1, 0, 0, \dots\}$ *then*

$$A_{k,l}(\alpha_1, \alpha_2, \dots) = \sum_{m=0}^{l} \binom{l}{m} A_{k-l,l-m}(\alpha_2, \alpha_3, \dots)\alpha_1^m \tag{1.11}$$

and conversely

$$A_{k,l}(\alpha_2, \alpha_3, \dots) = \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} A_{k+m,m}(\alpha_1, \alpha_2, \dots)\alpha_1^{l-m}. \tag{1.11'}$$

EXAMPLE 4. Bernoulli numbers $B_k^{(-l)}$ of order $-l \leqq 0$ are defined by

$$(e^w - 1)^l = \sum_{k=0}^{\infty} \frac{B_k^{(-l)}}{k!} w^{k+l}, \qquad l \geqq 0.$$

Applying (1.11′) to Example 2 we obtain

$$\frac{B_{k-l}^{(-l)}}{(k-l)!} = A_{k,l}\left(1, \frac{1}{2!}, \dots\right) = \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \frac{m^k}{k!}.$$

LEMMA D. *Let* $\omega = \alpha(w) = \sum_{n=1}^{\infty} \alpha_n w^n$, $\beta(\omega) = \sum_{n=1}^{\infty} \beta_n \omega^n$ *and* $\gamma(w) = \beta(\alpha(w)) = \sum_{n=1}^{\infty} \gamma_n w^n$. *Then*

$$A_{k,l}(\gamma) = \sum_{n=l}^{k} A_{k,n}(\alpha) A_{n,l}(\beta), \tag{1.12}$$

and in particular

$$\gamma_k = A_{k,1}(\gamma) = \sum_{n=0}^{k} A_{k,n}(\alpha)\beta_n \qquad (\beta_0 = 0,\ A_{k,0}(\alpha) = \delta_{k,0}). \tag{1.13}$$

Every analytic function $f(z)$ in a domain $D \subset \mathbf{C}$ defines a sequence $\alpha_n = f^{(n)}(z)/n!$, $n \geqq 1$, $z \in D$, so that $\alpha(w) = f(z+w) - f(z)$, and we have the corresponding Bell polynomials

$$A_{k,l}(f, z) = A_{k,l}(\alpha) = A_{k,l}\left(f'(z), \frac{f''(z)}{2!}, \dots\right).$$

Lemma D may be reformulated for that sequence:

LEMMA D′. *Let* $\zeta = f(z)$ *be analytic in* D *and* $g(\zeta)$ *in* $f(D)$. *Then* (cf. Jabotinsky [10]):

$$A_{k,l}(g \cdot f, z) = \sum_{n=l}^{k} A_{k,n}(f, z) A_{n,l}(g, f(z)). \tag{1.12'}$$

In particular we get Faa'-di-Bruno's formula ([5] Section 3.4)

$$\frac{1}{k!}(g \cdot f)^{(k)}(z) = \sum_{n=0}^{k} A_{k,n}(f, z)\frac{g^{(n)}(f(z))}{n!}. \tag{1.13'}$$

The following lemma has been proven by Jabotinsky in the case that $z = 0$, but its generalization is straightforward.

LEMMA E. *Let f be analytic and univalent near z, i.e., $f'(z) \neq 0$. Then*

$$A_{l,n}(g, \zeta) = \frac{n}{l} A_{-n,-l}(f, z), \qquad l \geqq n, \quad l \neq 0 \tag{1.14}$$

where $g = f^{-1}$ near $\zeta = f(z)$.

Finally we need a formula for Bell polynomials of negative degrees:

LEMMA F. *Suppose $\alpha_1 \neq 0$. For $l > 0$ we have*

$$A_{k-l,-l}(\alpha) = \sum_{n=0}^{k} (-1)^n \binom{l+n-1}{n} \alpha_1^{-l-n} A_{k,n}(\alpha_2, \alpha_3, \ldots). \tag{1.15}$$

PROOF. Since we have

$$w^l \alpha(w)^{-l} = \left(\alpha_1 + \sum_{n=1}^{\infty} \alpha_{n+1} w^n\right)^{-l} = \sum_{k=0}^{\infty} A_{k-l,-l}(\alpha) w^k,$$

we deduce

$$A_{k-1,-l}(\alpha) = \frac{1}{k!}\frac{d^k}{dw^k}\left(\alpha_1 + \sum_{n=1}^{\infty} \alpha_{n+1} w^n\right)^{-l}_{w=0} = \frac{(g \cdot f)^{(k)}(0)}{k!}$$

where

$$f(w) = \sum_{n=1}^{\infty} \alpha_{n+1} w^n, \qquad g(\omega) = (\alpha_1 + \omega)^{-l}.$$

Thus (1.13′) implies (1.15). Q.E.D.

2. Aharonov invariants

For a sequence $\alpha = \{\alpha_n\}_1^\infty$ let

$$\log\left(1 + \sum_{n=1}^{\infty} \alpha_n w^n\right) = \sum_{k=1}^{\infty} A_k(\alpha) w^k \tag{2.1}$$

and

$$\left(1 + \sum_{n=1}^{\infty} \alpha_n w^n\right)^{-1} = 1 - \sum_{k=1}^{\infty} \psi_k(\alpha) w^k. \tag{2.2}$$

(Notice that

$$\log\left(1-\sum_{n=1}^{\infty}\psi_n(\alpha)w^n\right)=-\sum_{k=1}^{\infty}A_k(\alpha)w^k\right). \tag{2.3}$$

In particular if $f(z)$ is analytic and univalent near z, let

$$\alpha_n=\frac{f^{(n+1)}(z)}{(n+1)!\,f'(z)}$$

and

$$A_k(\alpha)=A_k(f,z),\qquad \psi_k(\alpha)=\psi_k(f,z),$$

i.e.,

$$\log\left(1+\sum_{n=1}^{\infty}\frac{f^{(n+1)}(z)}{(n+1)!\,f'(z)}w^n\right)=\log\frac{f(z+w)-f(z)}{wf'(z)}=\sum_{k=1}^{\infty}A_k(f,z)w^k \tag{2.1'}$$

and

$$\frac{f'(z)}{f(z+w)-f(z)}-\frac{1}{w}=-\sum_{n=0}^{\infty}\psi_{n+1}(f,z)w^n \qquad \text{(cf. Aharonov [2]).} \tag{2.2'}$$

LEMMA 1. (i) *For a sequence* $\alpha=\{\alpha_n\}_1^{\infty}$ *we have*

$$A_k(\alpha)=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n}A_{k,n}(\alpha), \tag{2.4}$$

$$\psi_k(\alpha)=\sum_{n=1}^{k}(-1)^{n-1}A_{k,n}(\alpha). \tag{2.5}$$

(ii) *If* $f(z)$ *is analytic with* $f'(z)\neq 0$, *then*

$$A_k(f,z)=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n}\binom{k}{n}f'(z)^{-n}A_{k+n,n}(f,z) \tag{2.4'}$$

and

$$\psi_k(f,z)=\sum_{n=1}^{k}(-1)^{n-1}\binom{k+1}{n+1}f'(z)^{-n}A_{k+n,n}(f,z). \tag{2.5'}$$

PROOF. Formulas (2.4) and (2.5) follow by applying (1.13) to $\omega=\alpha(w)=\sum_{n=1}^{\infty}\alpha_n w^n$, with

$$\beta(\omega)=\log(1+\omega)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\omega^n \quad\text{or}\quad \beta(\omega)=\frac{\omega}{1+\omega}=\sum_{n=1}^{\infty}(-1)^{n-1}\omega^n,$$

respectively. Identities (1.5), (1.11') and (1.9) then yield (2.4') and (2.5') as follows:

$$A_k(f,z)=\sum_{m=1}^{k}\frac{(-1)^{m-1}}{m}A_{k,m}\left(\frac{f''(z)}{2f'(z)},\frac{f'''(z)}{3!f'(z)},\dots\right)$$

$$=\sum_{m=1}^{k}\frac{(-1)^{m-1}}{m}f'(z)^{-m}\sum_{n=1}^{m}(-1)^{m-n}\binom{m}{n}A_{k+n,n}\left(f'(z),\frac{f''(z)}{2!},\dots\right)f'(z)^{m-n}$$

$$=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n}f'(z)^{-n}A_{k+n,n}(f,z)\sum_{m=n}^{k}\binom{m-1}{n-1}$$

$$=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n}\binom{k}{n}f'(z)^{-n}A_{k+n,n}(f,z),$$

and

$$\psi_k(f,z)=\sum_{m=1}^{k}(-1)^{m-1}A_{k,m}\left(\frac{f''}{2!f'},\frac{f'''}{3!f'},\dots\right)$$

$$=\sum_{m=1}^{k}(-1)^{m-1}f'(z)^{-m}\sum_{n=1}^{m}(-1)^{m-n}\binom{m}{n}A_{k+n,n}(f,z)f'(z)^{m-n}$$

$$=\sum_{n=1}^{k}(-1)^{n-1}f'(z)^{-n}A_{k+n,n}(f,z)\sum_{m=n}^{k}\binom{m}{n}$$

$$=\sum_{n=1}^{k}(-1)^{n-1}\binom{k+1}{n+1}f'(z)^{-n}A_{k+n,n}(f,z). \qquad \text{Q.E.D.}$$

EXAMPLE 5. Let $f(z)=e^z$. Then

$$\log\left(\frac{f(z+w)-f(z)}{wf'(z)}\right)=\log\left(\frac{e^w-1}{w}\right)=\sum_{k=1}^{\infty}A_k(e^z,z)w^k$$

and

$$\frac{f'(z)}{f(z+w)-f(z)}-\frac{1}{w}=\frac{1}{e^w-1}-\frac{1}{w}=-\sum_{k=1}^{\infty}\psi_k(e^z,z)w^{k-1},$$

i.e., $A_k(e^z,z)$ and $\psi_k(e^z,z)$ are independent of z. Moreover, by the definition of Bernoulli numbers B_k we now have

$$\frac{e^w}{e^w-1}-\frac{1}{w}=\frac{d}{dw}\left[\log\frac{e^w-1}{w}\right]=\sum_{k=1}^{\infty}kA_k(e^z,z)w^{k-1}$$

$$=1+\frac{1}{e^w-1}-\frac{1}{w}=1-\sum_{k=1}^{\infty}\psi_k(e^z,z)w^{k-1}=1+\sum_{k=1}^{\infty}\frac{B_k}{k!}w^{k-1}.$$

Thus, from (2.4′) and (2.5′) we deduce explicit formulas for Bernoulli numbers (cf. [6]):

$$
(2.6)\quad \begin{cases} 1+B_1 = A_1(e^z, z) = A_{2,1}\left(\left\{\dfrac{1}{n!}\right\}_1^\infty\right) = B_1^{(-1)} = \frac{1}{2}, \\[2ex] B_k = k\cdot k!\,A_k(e^z, z) = \displaystyle\sum_{m=1}^{k} \frac{k(-1)^{m-1}}{m}\binom{k}{m} B_k^{(-m)}, \qquad k \geqq 2 \\[2ex] \text{or} \\[2ex] B_k = -k!\,\psi_k(e^z, z) = \displaystyle\sum_{m=1}^{k} (-1)^m \binom{k+1}{m+1} B_k^{(-m)}, \qquad k \geqq 1. \end{cases}
$$

REMARK. Aharonov has already shown that (see [2]):

$$
(2.7)\qquad \psi_k(g\cdot f, z) = \psi_k(f, z), \qquad k \geqq 2
$$

for every Möbius transformation g. Similarly one can show that:

$$
(2.7')\qquad A_k(h\cdot f, z) = A_k(f, z), \qquad k \geqq 1
$$

for every affine mapping $h(z) = az + b$, $a \neq 0$.

3. Generalized Grunsky coefficients

For a given sequence $\alpha = \{\alpha_n\}_1^\infty$ let

$$
(3.1)\qquad \log(1 - t\zeta - \alpha(\zeta)) = -\sum_{n=1}^{\infty} \frac{F_n(\alpha; t)}{n}\zeta^n, \quad \alpha(\zeta) = \sum_{n=1}^{\infty} \alpha_n \zeta^n.
$$

Then by (2.1), (2.4) and (1.10)

$$
\begin{aligned}
F_n(\alpha; t) &= -nA_n(-t-\alpha_1, -\alpha_2, -\alpha_3, \ldots) \\
&= \sum_{k=1}^{n} (-1)^k \frac{n}{k} A_{n,k}(-t-\alpha_1, -\alpha_2, \ldots) \\
&= \sum_{k=1}^{n} \frac{n}{k} \sum_{m=0}^{k} \binom{k}{m} A_{n-m,k-m}(\alpha) t^m \\
&= \sum_{m=0}^{n} F_{n,m}(\alpha) t^m
\end{aligned}
$$

where

$$
(3.2)\qquad F_{n,m}(\alpha) = \begin{cases} -nA_n(-\alpha) = \displaystyle\sum_{k=1}^{n} \frac{n}{k} A_{n,k}(\alpha) & \text{for } m = 0, \\[2ex] \dfrac{n}{m} \displaystyle\sum_{k=m}^{n} \binom{k-1}{m-1} A_{n-m,k-m}(\alpha) & \text{for } n \geqq m > 0. \end{cases}
$$

The polynomials $F_n(\alpha; t)$ are Faber polynomials associated with the sequence α. Gunsky coefficients $b_{k,n}(\alpha)$ are defined by the generating function

$$\log \frac{\tilde{\alpha}(\zeta)-\tilde{\alpha}(\omega)}{\zeta-\omega} = -\sum_{k,n=1}^{\infty} b_{k,n}(\alpha) w^{-k}\zeta^{-n} \tag{3.3}$$

where $\tilde{\alpha}(z) = z - z\alpha(z^{-1})$, which yields at once

LEMMA 2. (i) $b_{k,n}(\alpha) = b_{n,k}(\alpha)$.

(ii) $b_{k,n}(\alpha) = b_{k,n}(\alpha_2, \alpha_3, \ldots)$ *are independent of* α_1,

(iii) *Formulas* (3.1) *and* (3.3) *imply*

$$F_n\left(\alpha; \frac{1-\alpha(\omega)}{\omega}\right) = \omega^{-n} + n\sum_{k=1}^{\infty} b_{k,n}(\alpha)\omega^k. \tag{3.4}$$

For example, if $F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$ is analytic in $|z|>1$, its ordinary Grunsky coefficients are exactly Grunsky coefficients associated with the sequence $\alpha_n = -b_n$.

LEMMA 3. *For a given sequence* $\alpha = \{\alpha_n\}_1^{\infty}$ *we have*

$$nb_{k,n}(\alpha) = \sum_{m=0}^{n} F_{n,m}(\alpha) C_{k,m}(\alpha) \tag{3.5}$$

where

$$C_{k,m}(\alpha) = \sum_{l=0}^{m} (-1)^l \binom{m}{l} A_{k+m,l}(\alpha). \tag{3.6}$$

PROOF. Evaluate the left-hand side of (3.4) as follows:

$$\begin{aligned} F_n\left(\alpha; \frac{1-\alpha(\omega)}{\omega}\right) &= \sum_{m=0}^{n} F_{n,m}(\alpha)\omega^{-m}(1-\alpha(\omega))^m \\ &= \sum_{m=0}^{n} F_{n,m}(\alpha) \sum_{l=-m}^{\infty} C_{lm}(\alpha)\omega^l \end{aligned}$$

where, by (1.13'),

$$C_{k-m,m}(\alpha) = \frac{1}{k!}\frac{d^k}{d\omega^k}(1-\alpha(\omega))^m \Big|_{\omega=0} = \sum_{l=0}^{m}(-1)^l\binom{m}{l} A_{k,l}(\alpha), \qquad k \geqq m$$

and thus

$$\begin{aligned} F_n\left(\alpha; \frac{1-\alpha(\omega)}{\omega}\right) &= \sum_{m=0}^{n} F_{n,m}(\alpha)\left\{\sum_{l=-m}^{0} C_{l,m}(\alpha)\omega^l + \sum_{l=1}^{\infty} C_{l,m}(\alpha)\omega^l\right\} \\ &= \sum_{l=-n}^{0}\left\{\sum_{m=-l}^{n} F_{n,m}(\alpha)C_{l,m}(\alpha)\right\}\omega^l + \sum_{l=1}^{\infty}\left\{\sum_{m=0}^{n} F_{n,m}(\alpha)C_{l,m}(\alpha)\right\}\omega^l. \end{aligned}$$

Comparing this to the right-hand side of (3.4), (3.5) immediately follows.

Q.E.D.

REMARK. From the proof it also follows that

$$\sum_{m=k}^{n} F_{n,m}(\alpha)C_{-k,m}(\alpha)=\delta_{k,n}, \qquad 0 \leqq k \leqq n. \tag{3.7}$$

For example, by formula (3.2) we have:

$$F_{n,n}(\alpha)=1, \qquad F_{n,n-1}(\alpha)=n\alpha_1,$$

$$F_{n,n-2}(\alpha)=n\alpha_2+\binom{n}{2}\alpha_1^2,$$

$$F_{n,n-3}(\alpha)=n\alpha_3+n(n-2)\alpha_1\alpha_2+\binom{n}{3}\alpha_1^3.$$

Also,

$$C_{k,0}(\alpha)=0 \qquad (\text{for } k>0),$$

$$C_{k,1}(\alpha)=-\alpha_{k+1},$$

$$C_{k,2}(\alpha)=-2\alpha_{k+2}+\sum_{n=1}^{k+1}\alpha_n\alpha_{k-n+2}.$$

Thus,

$$b_{k,1}(\alpha)=-\alpha_{k+1} \quad \text{and} \quad b_{k,2}(\alpha)=-\alpha_{k+2}+\frac{1}{2}\sum_{n=2}^{k}\alpha_n\alpha_{k-n+2}. \tag{3.8}$$

Now let $f(z)$ be an analytic function in a domain $D \subset \mathbf{C}$ and $z \in D\setminus\{\infty\}$. Grunsky coefficients of f at z are defined by means of the generating function

$$\log\frac{f(z+\zeta)-f(z+\omega)}{\zeta-\omega}=-\sum_{k,n=0}^{\infty} b_{k,n}(f,z)\zeta^k\omega^n. \tag{3.9}$$

In particular, for $\omega=0$ we get, by (2.1′),

$$\log\frac{f(z+\zeta)-f(z)}{\zeta}=-\sum_{k=0}^{\infty} b_{k,0}(f,z)\zeta^k$$

$$=\log f'(z)+\sum_{k=1}^{\infty} A_k(f,z)\zeta^k,$$

i.e.,

$$b_{k,0}(f,z)=\begin{cases}-\log f'(z) & \text{for } k=0,\\ -A_k(f,z) & \text{for } k\geqq 1.\end{cases} \tag{3.10}$$

Hence,

$$\begin{aligned}&\log\frac{f(z+\zeta)-f(z+\omega)}{\zeta-\omega}-\log\frac{f(z+\zeta)-f(z)}{\zeta f'(z)}-\log\frac{f(z+\omega)-f(z)}{\omega f'(z)}-\log f'(z)\\ &=\log\left\{\left[\frac{f'(z)}{f(z+\zeta)-f(z)}-\frac{f'(z)}{f(z+\omega)-f(z)}\right]\Big/(\zeta^{-1}-\omega^{-1})\right\}\\ &=-\sum_{k,n=1}^{\infty}b_{k,n}(f,z)\zeta^k\omega^n.\end{aligned} \tag{3.11}$$

But since we have by (2.2′)

$$\frac{f'(z)}{f(z+w)-f(z)}=\frac{1}{w}-\sum_{n=1}^{\infty}\psi_n(f,z)w^{n-1}=\tilde{\alpha}(w^{-1})=\frac{1-\alpha(w)}{w}$$

it appears that Grunsky coefficients of f at z are exactly Grunsky coefficients of the sequence $\{\psi_n(f,z)\}_{n=1}^{\infty}$. Hence,

LEMMA 2′. (i) $b_{k,n}(f,z)=b_{n,k}(f,z)$.

(ii) *For all k, $n\geqq 1$, $b_{k,n}(f,z)=b_{k,n}(\psi_2,\psi_3,\dots)$ are independent of $\psi_1(f,z)$ and therefore by* (2.7)

$$b_{k,n}(g\cdot f,z)=b_{k,n}(f,z) \quad \textit{for all Möbius transformations } g. \tag{3.12}$$

(iii) *If $F_n(f,z;t)$ is Faber polynomial of degree n for the sequence $\{\psi_n(f,z)\}_{n=1}^{\infty}$, then*

$$F_n\left(f,z;\frac{f'(z)}{f(z+\zeta)-f(z)}\right)=\zeta^{-n}+n\sum_{k=1}^{\infty}b_{k,n}(f,z)\zeta^k, \qquad n\geqq 1. \tag{3.4′}$$

THEOREM 1. *Let f be analytic at z.*

(i) *For $n\geqq 1$ we have*

$$b_{m,n}(f,z)=\frac{1}{n}\sum_{l=1}^{n}(-1)^l\binom{m+l-1}{m}\frac{\psi_{m+l}^{(n-l)}(f,z)}{(n-l)!}. \tag{3.13}$$

(ii) *If $F_n(f,z;t)=\sum_{m=0}^{n}F_{n,m}(f,z)t^m$, then*

$$f'(z)^mF_{n,m}(f,z)=\begin{cases}A_{-m,-n}(f^{-1},f(z)), & 0\leqq m\leqq n,\\ (n/m)A_{n,m}(f,z), & 1\leqq m\leqq n,\end{cases} \tag{3.14}$$

and

$$(3.15)\qquad b_{k,n}(f,z)=\begin{cases}\dfrac{1}{n}\displaystyle\sum_{m=1}^{n} A_{k,-m}(f,z)A_{-m,-n}(f^{-1},f(z)),\\[2ex] \displaystyle\sum_{m=1}^{n}\frac{1}{m}A_{k,-m}(f,z)A_{n,m}(f,z).\end{cases}$$

PROOF. (i) According to formulas (3.9) and (2.2′), we have for $n \geqq 1$, $m \geqq 0$:

$$\begin{aligned} b_{m,n}(f,z) &= -\frac{1}{m!\,n!}\frac{\partial^{m+n}}{\partial\zeta^n\partial\omega^m}\log\frac{f(z+\zeta)-f(z+\omega)}{(z+\zeta)-(z+\omega)}\Bigg|_{\zeta=\omega=0} \\ &= -\frac{1}{m!\,n!}\frac{\partial^{m+n-1}}{\partial\zeta^{n-1}\partial\omega^m}\left\{\frac{f'(\zeta)}{f(\zeta)-f(\omega)}-\frac{1}{\zeta-\omega}\right\}\Bigg|_{\zeta=\omega=z} \\ &= -\frac{1}{m!\,n!}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}\left\{\sum_{k=m+1}^{\infty}\frac{(k-1)!}{(k-m-1)!}\psi_k(f,\zeta)(\omega-\zeta)^{k-m-1}\right\}\Bigg|_{\zeta=\omega=z} \\ &= \frac{1}{n!}\sum_{k=m+1}^{\infty}(-1)^{k-m}\binom{k-1}{m}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}\{\psi_k(f,\zeta)(\zeta-z)^{k-m-1}\}\Bigg|_{\zeta=z} \\ &= \frac{1}{n}\sum_{k=m+1}^{m+n}(-1)^{k-m}\binom{k-1}{m}\frac{\psi_k^{(m+n-k)}(f,z)}{(m+n-k)!}. \end{aligned}$$

(ii) Evaluating the left-hand side of (3.4′) we get

$$\begin{aligned} &F_n\left(f,z;\frac{f'(z)}{f(z+\alpha)-f(z)}\right) \\ &= \sum_{m=0}^{n} F_{n,m}(f,z)f'(z)^m(f(z+\zeta)-f(z))^{-m} \\ &= \sum_{m=0}^{n} F_{n,m}(f,z)f'(z)^m\left\{\sum_{l=-m}^{0}A_{l,-m}(f,z)\zeta^l+\sum_{l=1}^{\infty}A_{l,-m}(f,z)\zeta^l\right\} \\ &= \sum_{l=-n}^{0}\left\{\sum_{m=-l}^{n}A_{l,-m}(f,z)F_{n,m}(f,z)f'(z)^m\right\}\zeta^l \\ &\quad+\sum_{l=1}^{\infty}\left\{\sum_{m=0}^{n}A_{l,-m}(f,z)F_{n,m}(f,z)f'(z)^m\right\}\zeta^l. \end{aligned}$$

Comparing it to the right-hand side of (3.4′) we conclude

$$(3.16)\qquad \sum_{m=k}^{n} A_{-k,-m}(f,z)F_{n,m}(f,z)f'(z)^m=\delta_{k,n},\qquad 0\leqq k\leqq n,$$

and

$$(3.17)\qquad \sum_{m=1}^{n} A_{k,-m}(f,z)F_{n,m}(f,z)f'(z)^m=nb_{k,n}(f,z),\qquad k\geqq 1.$$

But by (1.12′) we have

$$\sum_{m=k}^{n} A_{-k,-m}(f,z)A_{-m,-n}(f^{-1},f(z)) = A_{-k,-n}(f^{-1}\cdot f,z) = \delta_{k,n}.$$

Thus (3.16) with (1.14) yield (3.14), and (3.14) with (3.17) and (1.14) imply (3.15).

Q.E.D.

COROLLARY 1. (i) *By* (3.10) *and* (3.13) *we have*

$$A_k(f,z) = -b_{0,k}(f,z) = \frac{1}{k}\sum_{m=1}^{k}(-1)^{m-1}\frac{\psi_m^{(k-m)}(f,z)}{(k-m)!}, \qquad k \geqq 1 \tag{3.18}$$

and

$$\psi_{n+1}(f,z) = -b_{1,n}(f,z) = \sum_{l=1}^{n}(-1)^{l-1}\frac{l}{n}\frac{\psi_{l+1}^{(n-l)}(f,z)}{(n-l)!}, \qquad k \geqq 1. \tag{3.19}$$

(ii) *Comparing* (3.13) *for* $n=2$:

$$b_{m,2}(f,z) = \frac{m+1}{2}\psi_{m+2}(f,z) - \tfrac{1}{2}\psi'_{m+1}(f,z)$$

with (3.8), *one gets* (cf. [1] formula 2.5(d)):

$$(m+3)\psi_{m+2}(f,z) = \psi'_{m+1}(f,z) + \sum_{n=2}^{m}\psi_n(f,z)\psi_{m-n+2}(f,z).$$

Next we generalize the transformation formula (2.4′) of [7]:

THEOREM 2. *Let* $g(z) = (az+b)/(cz+d)$, $ad-bc=1$. *Then*

$$b_{m,n}(f\cdot g,z) = \sum_{l=1}^{m}\sum_{k=1}^{n}\binom{m-1}{l-1}\binom{n-1}{k-1}(-c)^{m+n-l-k}g'(z)^{(m+n+l+k)/2}b_{l,k}(f,g(z)),$$

$$m,n \geqq 1. \tag{3.20}$$

PROOF. Denote

$$\Phi_f(z,\zeta) = \log\frac{f(z)-f(\zeta)}{z-\zeta}.$$

Then for every Möbius transformation one gets

$$\Phi_{f\cdot g}(z,\zeta) = \Phi_f(g(z),g(\zeta)) + \tfrac{1}{2}\log g'(z) + \tfrac{1}{2}\log g'(\zeta).$$

Hence, by formula (3.9) we may write

$$b_{m,n}(f\cdot g,z) = \frac{1}{m!\,n!}\frac{\partial^{m+n}}{\partial\xi^n\partial\omega^m}\Phi_f(g(\zeta),g(\omega))\Big|_{\zeta=\omega=z}, \qquad m,n\geqq 1.$$

Now use Faá-di-Bruno's formula (1.13') and the identity

$$A_{k,n}(g,z)=\binom{k-1}{n-1}\frac{(-c)^{k-n}}{(cz+d)^{k+n}}\qquad \text{for } g(z)=\frac{az+b}{cz+d},\quad ad-bc=1$$

(see [7]), and (3.20) follows. Q.E.D.

COROLLARY 2. *If f is univalent in the unit disc U, then*

(3.21) $$(1-|z|^2)^{m+n}|b_{m,n}(f,z)|\leqq q_{m-1}(|z|)q_{n-1}(|z|),\qquad |z|<1$$

where

$$q_k(x)=\sum_{l=0}^{k}\binom{k}{l}\frac{x^l}{\sqrt{k-l+1}},$$

and if f has a μ-quasiconformal extension to $\mathbf{C}$, *then*

(3.21') $$(1-|z|^2)^{m+n}|b_{m,n}(f,z)|\leqq q_{m-1}(|z|)q_{n-1}(|z|)\|\mu\|_\infty,\qquad |z|<1.$$

This corollary is a generalization of Theorem 1 in [7], and the proof is identical, using (3.20) and the Grunsky inequality

$$|b_{m,n}(f,0)|\leqq 1/\sqrt{mn}\qquad (\text{or } \|\mu\|_\infty/\sqrt{mn}, \text{ respectively}).$$

Since inequality (3.21) will be improved in the next section, we omit details of its proof.

4. Grunsky inequalities, generalization and consequences

THEOREM 3. *Let f be a univalent meromorphic function in a domain D of any connectivity. Then*

(4.1) $$\sum_{k=1}^{\infty}k\left|\sum_{n=1}^{\infty}d(z,\partial D)^{n+k}b_{k,n}(f,z)\lambda_n\right|^2\leqq\sum_{k=1}^{\infty}\frac{|\lambda_k|^2}{k},\qquad z\in D$$

for all sequences $\{\lambda_n\}$ of complex numbers for which the right-hand side of (4.1) *converges. If f has a μ-quasiconformal extension to* $\mathbf{C}$, *the right-hand side of* (4.1) *is multiplied by* $\|\mu\|_\infty^2<1$.

PROOF. According to formula (3.11), $b_{k,n}(f,z)$ are the ordinary Grunsky coefficients of the univalent function

$$F\left(z;\frac{1}{\zeta}\right)=\frac{f'(z)}{f(z+\zeta^{-1})-f(z)}=\zeta-\sum_{n=1}^{\infty}\psi_n(f,z)\zeta^{-n+1},\qquad |\zeta|>d(z,\partial D)^{-1}.$$

Thus (4.1) is just the classical Grunsky inequality for $F(z;1/\zeta)$ (see the proof in [11]). Q.E.D.

In particular, if $\lambda_n = \delta_{n,m}$ for some $m \geqq 1$, then

COROLLARY 3. *If f is univalent in D (with a μ-quasiconformal extension to* **C**), *then for every $z \in D$:*

$$(4.2) \qquad \sum_{k=1}^{\infty} kd(z,\partial D)^{2(m+k)}|b_{k,m}(f,z)|^2 \leqq 1/m \quad (\leqq \|\mu\|_\infty^2/m, \text{ respectively}).$$

Note that for $m = 1$, (4.2) becomes the area inequality (cf. [2], [7] and [8]):

$$(4.2') \qquad \sum_{k=1}^{\infty} kd(z,\partial D)^{2(k+1)}|\psi_{k+1}(f,z)|^2 \leqq 1 \quad (\leqq \|\mu\|_\infty^2, \text{ respectively}).$$

Aharonov has already studied the generalized Grunsky coefficients $b_{k,n}(f,z)$ (see [3]), and discovered sharp upper bounds for $b_{n,n}(f,z)$ for functions in the class S. Here a more explicit form of his result is established and generalized for all $b_{k,n}(f,z)$ of all univalent meromorphic functions in the unit disc U.

THEOREM 4. *Let $f(z)$ be univalent meromorphic in U. Then*

$$(4.3) \quad (1-|z|^2)^{m+n}|b_{m,n}(f,z)| \leqq \left(\frac{p_{m-1}(|z|^2)p_{n-1}(|z|^2)}{m\cdot n}\right)^{1/2}, \quad |z|<1, \ m,n \geqq 1,$$

where

$$(4.4) \qquad p_k(x) = \sum_{l=0}^{k} \binom{k+1}{l}\binom{k}{l} x^l = (k+1)\sum_{l=0}^{k}\binom{k}{l}^2 \frac{x^l}{k-l+1}$$

and

$$(4.5) \qquad \max_{0\leqq x\leqq 1} p_k(x) = p_k(1) = \binom{2k+1}{k}.$$

The proof of Theorem 4 depends on the following lemma.

LEMMA 4. *The polynomials $p_k(x)$, defined in* (4.4), *satisfy*

$$(4.6) \qquad \sum_{n=k}^{\infty} \binom{n}{k}\binom{n-1}{k-1} x^{n-k} = (1-x)^{-2k}p_{k-1}(x), \qquad k \geqq 1, \quad |x|<1,$$

e.g.,

$$\sum_{n=1}^{\infty} nx^{n-1} = (1-x)^{-2}, \quad \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)^2 x^{n-2} = \frac{1+2x}{(1-x)^4}, \ldots .$$

PROOF. Denote

$$Q_k(x) = k!(k-1)! \sum_{n=k}^{\infty} \binom{n}{k}\binom{n-1}{k-1} x^{n-k}, \qquad |x|<1.$$

Then

$$Q_k(x) = \frac{d}{dx}\left[x \frac{d}{dx} Q_{k-1}(x)\right], \qquad k = 2, 3, \dots$$

from which it follows by induction on k that

$$Q_k(x) = k!(k-1)!(1-x)^{-2k} p^*_{k-1}(x), \qquad k = 2, 3, \dots \tag{4.6*}$$

where $\{p^*_k(x)\}_0^\infty$ are polynomials satisfying the recursion formula

$$\begin{aligned} k(k+1)p^*_k(x) = {} & x(1-x)^2 p^{*\prime\prime}_{k-1}(x) + (1-x)(1+(4k-1)x)p^{*\prime}_{k-1}(x) \\ & + 2k(1+2kx)p^*_{k-1}(x) \end{aligned} \tag{4.7}$$

and since $Q_k(0) = k!(k-1)!$, $p^*_{k-1}(0) = 1$, it readily follows that

$$p^{*\prime}_k(0) = k(k+1).$$

Further differentiations of (4.7) yield at $x = 0$:

$$\begin{aligned} & k(k+1)p^{*(l)}_k(0) \\ & = (l+1)p^{*(l+1)}_{k-1}(0) + 2((2l+1)k - l^2)p^{*(l)}_{k-1}(0) + l(2k-l+1)^2 p^{*(l-1)}_{k-1}(0), \quad l \geqq 1 \end{aligned}$$

and this implies by induction

$$\frac{1}{l!} p^{*(l)}_k(0) = \binom{k+1}{l}\binom{k}{l}, \qquad 0 \leqq l \leqq k,$$

i.e., $p^*_k(x) = p_k(x)$ is the polynomial given in (4.4). If we set $x = 1$ at (4.7), identity (4.5) follows at once. Q.E.D.

PROOF OF THEOREM 4. By Lemma 2′(ii) we may assume, without loss of generality, that $f(z) = z + a_2 z^2 + \cdots \in S$. Then

$$F(0;\zeta) = \frac{f'(0)}{f(\zeta) - f(0)} = f(\zeta)^{-1}.$$

Hence, the generating function of $b_{k,n}(f,0) = b_{k,n}$ is

$$P(\zeta,\omega) = \log \frac{f(\zeta)^{-1} - f(\omega)^{-1}}{\zeta^{-1} - \omega^{-1}} = -\sum_{k,n=1}^{\infty} b_{k,n}\zeta^n\omega^n = \sum_{k=1}^{\infty} B_k(f,\zeta)\omega^k$$

where

$$B_k(f,\zeta)=\sum_{n=1}^{\infty} b_{k,n}(f,0)\zeta^n.$$

Using Grunsky inequalities Aharonov deduced (see [3])

$$\sum_{k=1}^{\infty} k\,|B_k^{(n)}(f,\zeta)|^2=\sum_{k=1}^{\infty} k\left|\sum_{m=n}^{\infty}\frac{m!}{(m-n)!}b_{k,m}\zeta^{m-n}\right|^2\leqq Q_n(|\zeta|^2),\quad |\zeta|<1 \tag{4.8}$$

where $Q_k(x)$ is given in the proof of Lemma 4. Now, by the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}\left|\frac{\partial^{m+n}}{\partial\zeta^n\partial\omega^m}P(\zeta,\omega)\right| &= \left|\sum_{k=m}^{\infty}\frac{k!}{(k-m)!}B_k^{(n)}(f,\zeta)\omega^{k-m}\right|\\ &\leqq\left\{\sum_{k=m}^{\infty}\frac{k!}{(k-m)!}\,\frac{(k-1)!}{(k-m)!}|\omega|^{2k-2m}\right\}^{1/2}\left\{\sum_{k=1}^{\infty}k\,|B_k^{(n)}(f,\zeta)|^2\right\}^{1/2} \qquad (4.9)\\ &\leqq Q_m(|\omega|^2)^{1/2}Q_n(|\omega|^2)^{1/2}.\end{aligned}$$

Next, by formula (3.9) we have for $m,n\geqq 1$ and $|z|<1$:

$$\begin{aligned}-m!\,n!\,b_{m,n}(f,z) &= \frac{\partial^{m+n}}{\partial\zeta^n\partial\omega^m}\log\frac{f(\zeta)-f(\omega)}{\zeta-\omega}\bigg|_{\zeta=\omega=z}\\ &=\frac{\partial^{m+n}}{\partial\zeta^n\partial\omega^m}\left[\log\frac{f(\zeta)^{-1}-f(\omega)^{-1}}{\zeta^{-1}-\omega^{-1}}+\log\frac{f(\zeta)}{\zeta}+\log\frac{f(\omega)}{\omega}\right]_{\zeta=\omega=z}\\ &=\frac{\partial^{m+n}}{\partial\zeta^n\partial\omega^m}P(\zeta,\omega)\Big|_{\zeta=\omega=z}.\end{aligned}$$

Thus inequality (4.9) implies

$$|b_{m,n}(f,z)|\leqq\frac{Q_m(|z|^2)^{1/2}Q_n(|z|^2)^{1/2}}{m!\,n!}.$$

This, with Lemma 4, completes the proof. Q.E.D.

REMARK. Inequality (4.3) improves (3.21), since we have

$$\frac{1}{m}p_{m-1}(x^2)=\sum_{l=0}^{m-1}\binom{m-1}{l}^2\frac{x^{2l}}{m-l}\leqq\left(\sum_{l=0}^{m-1}\binom{m-1}{l}\frac{x^l}{\sqrt{m-l}}\right)^2=q_{m-1}(x)^2.$$

In particular (4.3) implies an improvement on Theorem 1 of [7].

COROLLARY 4. *If $f(z)$ is univalent meromorphic in the unit disc U, then*

$$(4.10)\qquad \begin{aligned}(1-|z|^2)^n|\psi_n(f,z)| &\leqq \left(\sum_{k=0}^{n-2}\binom{n-2}{k}^2\frac{|z|^{2k}}{n-k-1}\right)^{1/2}\\ &\leqq \sqrt{\frac{1}{n-1}\binom{2n-3}{n-2}}, \qquad n\geqq 2, \quad |z|<1\end{aligned}$$

and

$$(4.11)\qquad \begin{aligned}(1-|z|^2)^{2n}|b_{n,n}(f,z)| &= (1-|z|^2)^{2n}\left|\sum_{l=1}^{n}(-1)^l\binom{n+l-1}{l}\frac{\psi_{n+l}^{(n-l)}(f,z)}{(n-l)!}\right|\\ &\leqq \sum_{k=0}^{n-1}\binom{n-1}{k}^2\frac{|z|^{2k}}{n-k}\\ &\leqq \frac{1}{n}\binom{2n-1}{n-1}, \qquad n\geqq 1, \quad |z|<1.\end{aligned}$$

As Aharonov has already pointed out, (4.11) is sharp, and equality is attained by the Koebe function $f(z)=z(1-z)^{-2}$.

References

1. D. Aharanov, *The Schwarzian derivative and univalent functions*, Doctoral Thesis, Technion, Israel Institute of Technology, 1967.

2. D. Aharonov, *A necessary and sufficient condition for univalence of a meromorphic function*, Duke Math. J. **36** (1969), 599–604.

3. D. Aharonov, *Sequence of necessary conditions for univalence of a meromorphic function*, Duke Math. J. **38** (1971), 595–598.

4. J. Burbea, *The Schwarzian derivative and the Poincaré metric*, Pac. J. Math. **85** (1979), 345–354.

5. L. Comtet, *Advanced Combinatorics, the Art of Finite and Infinite Expansions*, D. Reidel, Dordrecht, Boston, 1974.

6. H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly **79** (1972), 44–51.

7. R. Harmelin, *Aharonov invariants and univalent functions*, Isr. J. Math. **43** (1982), 244–254.

8. R. Harmelin, *Bergman kernel function and univalence criteria*, J. Analyse Math. **41** (1982), 249–258.

9. J. Hummel, *The Grunsky coefficients of a Schlicht function*, Proc. Amer. Math. Soc. **15** (1964), 142–150.

10. E. Jabotinsky, *Representation of functions by matrices, applications to Faber polynomials*, Proc. Amer. Math. Soc. **4** (1953), 546–553.

11. Ch. Pommerenke, *Univalent Functions*, Göttingen, 1975.

12. P. G. Todorov, *Explicit formulas for the coefficients of Faber polynomials with respect to univalent functions of the class* Σ, Proc. Amer. Math. Soc. **82** (1981), 431–438.